Intermodulation Noise in FM Systems Due to Transmission Deviations and AM / PM Conversion*

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Two noise contributors in FM systems are: (i) intermodulation noise due to transmission deviations; and (ii) intermodulation noise due to transmission deviations and AM/PM conversion, designated AM/PM intermodulation noise. Expressions for the second- and third-order AM/PM intermodulation noise are derived in terms of transmission medium coefficients and a continuous pre-emphasis characteristic, with the unpre-emphasized baseband signal being simulated by white Gaussian noise. These expressions have been programmed on a digital computer and representative noise responses and properties of AM/PM intermodulation noise were obtained. General properties and characteristics for the two noise contributors are documented in parallel for comparative purposes. It was found that AM/PM intermodulation noise can be a significant noise contributor in FM systems.

I. INTRODUCTION

Intermodulation noise is produced whenever a phase modulated signal is passed through a linear transmission medium whose amplitude and phase characteristics are nonlinear functions of frequency. The output signal from this medium is both envelope and phase modulated, with the phase modulation being a distorted replica of the input phase function. The envelope modulation and phase modulation functions are similar in that both consist of first (linear), second-, third-, and higher-order functions of the input phase function. They differ in that the coefficients of the terms making up the two modulating functions are related in different ways to the transmission medium characteristic.

The distortion terms higher than first order, in the output phase

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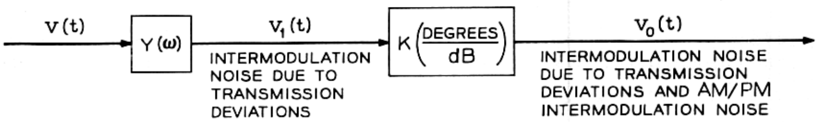
modulating function, produce intermodulation noise. This source of noise has been the subject of much work over the past ten to twenty years. The envelope distortion terms directly produce no degrading effects in linear systems. However, when the linear transmission medium is followed by a device that converts envelope variations at its input to phase variations at its output then a different noise-generating mechanism exists. This latter source of noise will be designated as "AM/PM intermodulation noise" to distinguish it from the intermodulation noise produced directly by transmission deviations.* The two phenomena are illustrated in Fig. 1 which depicts the two-step process involved in the



$$v(t) = \mathrm{EXP}^{i[\omega_c t + \varphi(t)]}$$
$$v_1(t) = \mathrm{EXP}^{a(t)}\ \mathrm{EXP}^{i[\omega_c t + \varphi_0(t)]}$$
$$v_0(t) = \mathrm{EXP}^{a_1(t)}\ \mathrm{EXP}^{i[\omega_c t + \varphi_0(t) + k\,a(t)]}$$

WHERE

$\varphi(t)$ = PHASE MODULATING FUNCTION DUE TO MULTICHANNEL SIGNAL

$\varphi_0(t) = \varphi(t)$ + PHASE DISTORTION TERMS

$k = 0.1516\,K$ = PHASE MODULATION INDEX IN RADIANS DIVIDED BY THE AMPLITUDE MODULATION INDEX

K = AM/PM CONVERSION CONSTANT MEASURED IN $\frac{\text{DEGREES}}{\text{dB}}$ ASSUMING $a(t) \ll 1$

$a_1(t) \neq a(t)$ IN GENERAL

$Y(\omega)$ = TRANSMISSION MEDIUM WITH TRANSMISSION DEVIATIONS

Fig. 1 — Model illustrating sources of intermodulation noise due to transmission deviations and AM/PM conversion.

AM/PM intermodulation noise generation. The AM/PM converter will be characterized by the constant K which has the dimension of degrees/dB and can be interpreted as the peak phase change at the output for a 1-dB change in envelope at the input. In reality, this K may be a function of a number of quantities, e.g., carrier drive power, frequency (carrier and/or baseband), bias levels, or may even be complex. However, many presently developed broadband radio systems use TWT amplifiers as power output tubes which are often the major source of AM/PM conversion within a radio repeater. These tubes, when driven at moder-

* Transmission deviations are defined as any deviation in the gain and phase characteristics from the ideal characteristics of constant gain and linear phase for all frequency components of the FM wave.

ate, essentially constant input power level and biased from well controlled sources, are adequately characterized for small envelope fluctuations by a constant K degrees/dB.[1]

Both noise phenomena are of prime interest in frequency modulated systems. Intermodulation noise due to transmission deviations is of interest because it is a recognized significant noise source. AM/PM intermodulation noise is of interest because of the basic lack of knowledge which has existed on this subject. Due to this deficiency, the AM/PM phenomenon has become the underlying scapegoat for many system problems that appear to be unexplainable using existing system knowledge.

The purpose of this paper is two-fold: (*i*) to present the mathematical development and ensuing solution for the problem of AM/PM intermodulation noise in FM systems; and (*ii*) to provide enough general information about the two noise contributors considered in this paper such that one can analyze a system's performance and/or set system requirements with some degree of confidence without having to necessarily utilize the associated digital computer programs.

The analysis to follow considers a linear transmission medium, with generalized transmission deviations, followed by an AM/PM converting device. The baseband signal is simulated by a Gaussian distributed band of noise with flat power density spectrum which is pre-emphasized by a continuous pre-emphasis function before the FM process. The end result of the treatment is the signal-to-noise ratio for second- and third-order AM/PM intermodulation noise. The mathematical framework for this paper is derived from a recent paper which treated the subject of intermodulation noise due to an imperfect transmission medium.[2] Certain facets of that work will be included here for the sake of continuity.

II. THEORY FOR AM/PM INTERMODULATION NOISE

2.1 *General Development*

Consider the system model shown in Fig. 1 where an FM signal is put into a linear transmission medium followed by an AM/PM converting device. The transfer function of the transmission medium is

$$Y(\omega) = \exp\left[-\alpha(\omega) - i\beta(\omega)\right] \tag{1}$$

and the impulse response is

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp(i\omega x)\,d\omega. \tag{2}$$

The FM signal input to the transmission medium is

$$v(t) = \exp \{i[\omega_c t + \varphi(t)]\} \tag{3}$$

and the output signal is

$$v_1(t) = \exp [a(t)] \exp \{i[\omega_c t + \varphi_o(t)]\}, \tag{4}$$

where ω_c is the carrier frequency and $\varphi(t)$ is the phase modulating signal. Since $Y(\omega)$ is a linear system, the input and output can be related by

$$v_1(t) = \int_{-\infty}^{\infty} v(t - x) g(x)\, dx. \tag{5}$$

Substituting (3) and (4) in (5) gives

$$\exp [a(t)] \exp [i\varphi_o(t)] = \int_{-\infty}^{\infty} \exp [i\varphi(t - x) - i\omega_c x] g(x)\, dx. \tag{6}$$

The output function, $\varphi_o(t)$, was the subject of a previous paper[2] and will not be considered further here. Our prime objective is to determine the envelope variation in terms of its functional relationship to the phase modulating signal, $\varphi(t)$. It follows from (6) that

$$a(t) = \operatorname{Re} \ln \int_{-\infty}^{\infty} \exp [i\varphi(t - x) - i\omega_c x] g(x)\, dx. \tag{7}$$

It can be shown that[2]

$$\begin{aligned} a(t) = {} & -\alpha(f_c) + m_{1i}\varphi' - \frac{m_{2i}}{2!}\varphi'' + \frac{m_{3i}}{3!}\varphi''' - \frac{m_{4i}}{4!}\varphi'''' + \cdots \\ & + \frac{l_{2r}}{2}\varphi'\varphi'' - \frac{l_{3r}}{6}\varphi'\varphi''' - \frac{\lambda_{2r}}{2}\varphi'^2 - \frac{l_{5r}}{8}\varphi''^2 + \cdots \\ & + \frac{l_{1i}}{4}\varphi'^2\varphi'' - \frac{\lambda_{3i}}{6}\varphi'^3 + \cdots + \frac{\lambda_{4r}}{24}\varphi'^4 \cdots , \end{aligned} \tag{8}$$

where the subscripts r and i denote the real and imaginary parts of the corresponding coefficients, and the prime notation indicates the derivative with respect to time. The argument of the phase functions in (8) is $t - t_d$ where t_d is an arbitrary delay.[2] The moments, m_n, in (8) are related to the transmission medium by

$$m_n = \frac{(-1)^n}{Y(\omega_c)} \left[\frac{d^n}{d(i\omega)^n} Y(\omega_c + \omega) \exp (i\omega t_d) \right]_{\omega=0} \tag{9}$$

and the l and λ coefficients are defined as follows:

$$l_1 = m_4 - 2m_1m_3 - m_2^2 + 2m_1^2m_2$$

$$l_2 = m_3 - m_1m_2$$

$$l_3 = m_4 - m_1m_3$$

$$l_5 = m_4 - m_2^2$$

$$\lambda_2 = m_2 - m_1^2$$

$$\lambda_3 = m_3 - 3m_1m_2 + 2m_1^3$$

$$\lambda_4 = m_4 - 4m_1m_3 - 3m_2^2 + 12\, m_1^2m_2 - 6m_1^4.$$

As an example, we have

$$l_{1i} = m_{4i} - 2m_{2r}m_{2i} - 2m_{1i}m_{3r} - 2m_{1i}^2m_{2i}$$

since $m_{1r} = 0$ (Appendix I of Ref. 2).
For the following transmission medium

$$\begin{aligned} Y(\omega + \omega_c) = [1 + g_1\omega + g_2\omega^2 + g_3\omega^3 + g_4\omega^4 \\ + \sum_{J=1}^{N} u_J \cos(P_J\omega + \theta_J)] \exp\Big\{i[b_2\omega^2 + b_3\omega^3 + b_4\omega^4 \\ + \sum_{J=1}^{N} \nu_J \sin(q_J\omega + \sigma_J)\Big\} \end{aligned} \tag{10}$$

the moments m_n given by (9) have been evaluated and expressed in terms of the transmission deviations in Appendix I of Ref. 2.

For the analysis to follow, the transmission deviations in (10) are limited to values typically encountered in broadband radio relay systems. However, the ripple type transmission deviations must have ripple periods greater than approximately twice the top baseband frequency. These restrictions are dictated by the limited number of terms of $a(t)$ which are to be considered.

Referring once again to Fig. 1, we see that when the output signal from $Y(\omega)$ passes through the AM/PM converter the envelope perturbation given by $a(t)$ is converted into a phase perturbation, given by $k\ a(t)$. The k coefficient is related to K (degrees/dB) as follows: the envelope distortion term expressed in dB is

$$20 \log \frac{\exp[a(t)]}{1} \text{ dB} = 8.686\, a(t) \text{ dB}$$

so the phase distortion, due to envelope perturbations, after AM/PM conversion is

$$8.686K \ a(t) \text{ degrees } \doteq 0.1516 \ K \ a(t) \text{ radians.}$$

We will let

$$k = 0.1516 \ K \text{ radians.}$$

Hence, the phase distortion function, due to envelope variations, after the AM/PM converting device is

$$k \ a(t) \text{ radians,} \tag{11}$$

where $a(t)$ is given by (8).

The analysis up to here has been perfectly general (except for the assumption that the AM/PM process may be represented by a constant factor). The terms in (8) consist of first- (linear), second-, third- and higher-order functions of the input phase modulating signal, $\varphi(t)$. The linear terms produce baseband amplitude distortion which we shall not concern ourselves with in this paper. Also, terms higher than third order will not be considered. This is not an undue restriction because the prime contributors of intermodulation type noise in broadband systems are second- and third-order phase distortion terms. Therefore, neglecting linear, fourth-, and higher-order terms in (8) gives*

$$\begin{aligned} k \, a(t) = k \left[\frac{l_{2r}}{2} \varphi' \varphi'' - \frac{l_{3r}}{6} \varphi' \varphi''' - \frac{\lambda_{2r}}{2} \varphi'^2 - \frac{l_{5r}}{8} \varphi''^2 \right] \\ + k \left[\frac{l_{1i}}{4} \varphi'^2 \varphi'' - \frac{\lambda_{3i}}{6} \varphi'^3 \right] \text{ radians.} \end{aligned} \tag{12}$$

Using the relationships

$$\frac{d}{dt} \varphi'^2 = 2\varphi' \varphi''$$

$$\frac{d^2}{dt^2} \varphi'^2 = 2\varphi' \varphi''' + 2\varphi''^2$$

$$\frac{d}{dt} \varphi'^3 = 3\varphi'^2 \varphi''$$

in (12) gives

$$k \ a(t) = \theta_2(t) + \theta_3(t) = \theta_T(t) \text{ radians,} \tag{13}$$

where

* It should be noted that additional second- and third-order terms exist which are not shown in (8) nor included in (12). These additional terms are considered to be negligible for the transmission deviation constraints previously mentioned.

$$\theta_2(t) = k\left[-\frac{\lambda_{2r}}{2} + \frac{l_{2r}}{4}\frac{d}{dt} - \frac{l_{3r}}{12}\frac{d^2}{dt^2}\right]\varphi'^2 + k\left[\frac{l_{4r}}{24}\right]\varphi''^2 \tag{14}$$

with

$$l_{4r} = 4l_{3r} - 3l_{5r} \tag{15}$$

and

$$\theta_3(t) = k\left[-\frac{\lambda_{3i}}{6} + \frac{l_{1i}}{12}\frac{d}{dt}\right]\varphi'^3. \tag{16}$$

In the Appendix it is shown that the second-order distortion, $\theta_2(t)$, and the third-order distortion, $\theta_3(t)$, are uncorrelated. Hence, the total AM/PM intermodulation noise power density spectrum, considering only second- and third-order distortions, is the sum of the two individual noise power density spectra.

2.2 *Second-Order Noise Power Density Spectrum*

In this section we will derive the equation for the second-order AM/PM intermodulation noise power density spectrum. The time representation for the second-order phase distortion due to AM/PM conversion was derived in the previous section and is

$$\theta_2(t) = k\left[-\frac{\lambda_{2r}}{2} + \frac{l_{2r}}{4}\frac{d}{dt} - \frac{l_{3r}}{12}\frac{d^2}{dt^2}\right]\varphi'^2 + k\left[\frac{l_{4r}}{24}\right]\varphi''^2. \tag{17}$$

The terms in brackets are operators on their respective functions, so (17) can be represented by the block diagram shown in Fig. 2, where

$$\frac{1}{k}G_1(\omega) = \left[\frac{l_{3r}}{12}\omega^2 - \frac{\lambda_{2r}}{2}\right] + i\left[\frac{l_{2r}}{4}\omega\right] \tag{18}$$

and

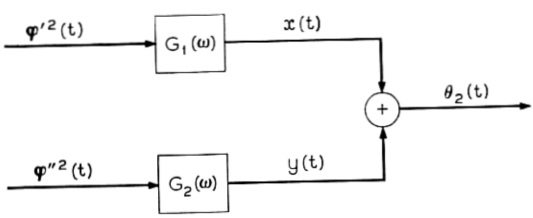


Fig. 2 — Second-order noise block diagram.

$$\frac{1}{k} G_2(\omega) = \frac{1}{24} l_{4r} . \tag{19}$$

It can easily be shown, using the relationship for the cross-correlation of linearly transformed random functions,[3] that the power density spectrum of $\theta_2(t)$ is

$$\begin{aligned} S_{\theta_2}(\omega) = {} & G_1(-\omega)\; G_1(\omega)\; S_{\varphi'^2}(\omega) + G_1(-\omega)\; G_2(\omega)\; S_{\varphi'^2\varphi''^2}(\omega) \\ & + G_2(-\omega)\; G_1(\omega)\; S_{\varphi''^2\varphi'^2}(\omega) + G_2(-\omega)\; G_2(\omega)\; S_{\varphi''^2}(\omega) \end{aligned} \tag{20}$$

where, for example, $S_{\varphi'^2\varphi''^2}(\omega)$ is the cross-power density spectrum of $\varphi'^2(t)$ and $\varphi''^2(t)$. As in Ref. 2, (20) can be expressed as

$$\begin{aligned} S_{\theta_2}(\omega) = {} & 2 \mid G_1(\omega) \mid^2 \mathfrak{F}[R_{\varphi'}{}^2(\tau)] + 2 \mid G_2(\omega) \mid^2 \mathfrak{F}[R_{\varphi''}{}^2(\tau)] \\ & + 2[G_1(-\omega)G_2(\omega) + G_2(-\omega)G_1(\omega)]\; \mathfrak{F}[R_{\varphi'\varphi''}{}^2(\tau)], \end{aligned} \tag{21}$$

where $G_1(\omega)$ and $G_2(\omega)$ are given by (18) and (19), respectively, and $\mathfrak{F}$ stands for the Fourier transform.

Now, redefining the transfer functions given in (18) and (19) we can write

$$\begin{aligned} \frac{1}{k^2} S_{\theta_2}(\omega) = {} & 2 \mid G_1(\omega) \mid^2 \mathfrak{F}[R_{\varphi'}{}^2(\tau)] + 2 \mid G_2(\omega) \mid^2 \mathfrak{F}[R_{\varphi''}{}^2(\tau)] \\ & + 2[G_1(-\omega)G_2(\omega) + G_2(-\omega)G_1(\omega)]\; \mathfrak{F}[R_{\varphi'\varphi''}{}^2(\tau)] \end{aligned} \tag{22}$$

where now

$$G_1(\omega) = \left[\frac{l_{3r}}{12}\,\omega^2 - \frac{\lambda_{2r}}{2}\right] + i\left[\frac{l_{2r}}{4}\,\omega\right] \tag{23}$$

$$G_2(\omega) = \frac{1}{24} l_{4r} . \tag{24}$$

Equation (22) is the second-order AM/PM intermodulation noise power density spectrum weighted by the AM/PM conversion parameter. The ability to pull the k out of the calculation provides great flexibility.

2.3 *Third-Order Noise Power Density Spectrum*

The time representation for the third-order phase distortion due to AM/PM conversion is, from (16),

$$\theta_3(t) = k\left[-\frac{\lambda_{3i}}{6} + \frac{l_{1i}}{12}\frac{d}{dt}\right]\varphi'^3 \tag{25}$$

which can be represented by the block diagram in Fig. 3 where

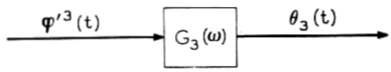


Fig. 3 — Third-order noise block diagram.

$$\frac{1}{k}\,G_3(\omega) = \left[-\frac{\lambda_{3i}}{6}\right] + i\left[\frac{l_{1i}}{12}\,\omega\right]. \tag{26}$$

It follows that the third-order AM/PM intermodulation noise power density spectrum is

$$S_{\theta_3}(\omega) = |\,G_3(\omega)\,|^2\, S_{\varphi'^3}(\omega).$$

It can be shown that[6]

$$S_{\varphi'^3}(\omega) = 6\,\mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)] + 9R_{\varphi'}^{\ 2}(0)\; S_{\varphi'}(\omega), \tag{27}$$

which can be written

$$S_{\varphi'^3}(\omega) = 6\mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)] \tag{28}$$

since $9\,R_{\varphi'}^{\ 2}(0)\; S_{\varphi'}(\omega)$ is a scaled power density spectrum of the input FM signal and hence can be neglected since it does not contribute to the distortion.* Therefore,

$$S_{\theta_3}(\omega) = 6\,|\,G_3(\omega)\,|^2\, \mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)] \tag{29}$$

where $G_3(\omega)$ is given by (26). Redefining the transfer function we have

$$\frac{1}{k^2}\, S_{\theta_3}(\omega) = 6\,|\,G_3(\omega)\,|^2\, \mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)] \tag{30}$$

where now

$$G_3(\omega) = \left[-\frac{\lambda_{3i}}{6}\right] + i\left[\frac{l_{1i}}{12}\,\omega\right]. \tag{31}$$

Hence, (30) gives the third-order AM/PM intermodulation noise power density spectrum weighted by the AM/PM conversion parameter.

A quantity of interest in engineering problems is the signal-to-noise ratio. Thus, we now characterize the simulated multichannel baseband signal.

2.4 *Pre-Emphasized Signal Power Density Spectrum*

The basic block diagram arrangement for a typical signal transmission path is shown in Fig. 4. The unpre-emphasized baseband signal is ob-

* This term causes baseband amplitude distortion instead of intermodulation noise.

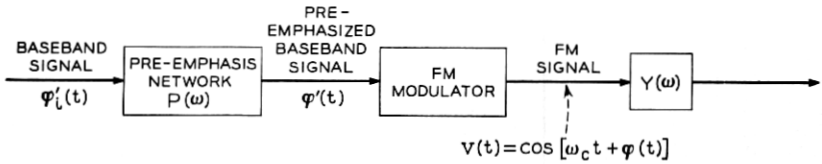


Fig. 4 — Typical signal transmission path.

tained from the frequency division multiplex terminals, directly or via a transmission facility, and is pre-emphasized prior to being applied to a FM modulator. The output of the FM modulator is consistent with $v(t)$ shown in Fig. 1. Assume that the unpre-emphasized baseband signal has a Gaussian distribution and a flat power density spectrum, P_o, between $-f_b$ and f_b, where f_b is the top baseband frequency. The output power density spectrum from the pre-emphasis network is

$$S_{\varphi'}(\omega) = P_o \mid P(\omega) \mid^2, \qquad \mid f \mid \leqq f_b \tag{32}$$

where $P(\omega)$ is the transfer function of the pre-emphasis network. Letting

$$\mid P(\omega) \mid^2 = a_0 + a_2 f^2 + a_4 f^4 + a_6 f^6, \qquad \mid f \mid \leqq f_b, \tag{33}$$

where the a's are real constants, we have

$$S_{\varphi'}(\omega) = P_o[a_0 + a_2 f^2 + a_4 f^4 + a_6 f^6], \qquad \mid f \mid \leqq f_b. \tag{34}$$

It can easily be shown that[2]

$$P_o = \frac{(2\pi\sigma)^2}{2f_b\left(a_0 + \dfrac{a_2 f_b^2}{3} + \dfrac{a_4 f_b^4}{5} + \dfrac{a_6 f_b^6}{7}\right)}, \ (\text{rad/sec})^2/\text{Hz} \tag{35}$$

where σ = rms frequency deviation, in Hz, due to the baseband signal, and f_b is in Hz. Equation (34) gives the power density spectrum of the pre-emphasized baseband signal in terms of the coefficients of a continuous pre-emphasis characteristic, and in terms of system parameters, σ and f_b.

2.5 *Signal-to-Noise Ratio*

We are now in a position to express the signal-to-noise ratio for second- and third-order AM/PM intermodulation noise. The expressions given in (22) and (30) are for PM distortions so we convert them to FM distortions by multiplying by ω^2. Hence, the signal-to-noise ratios can be expressed as

$$\left[10 \log \frac{S_{\varphi'}(\omega)}{\omega^2 S_{\theta_2}(\omega)}\right]_{\text{2nd order}} = 10 \log \frac{S_{\varphi'}(\omega)}{\left(\frac{\omega}{k}\right)^2 S_{\theta_2}(\omega)} - 20 \log k \tag{36}$$

and

$$\left[10 \log \frac{S_{\varphi'}(\omega)}{\omega^2 S_{\theta_3}(\omega)}\right]_{\text{3rd order}} = 10 \log \frac{S_{\varphi'}(\omega)}{\left(\frac{\omega}{k}\right)^2 S_{\theta_3}(\omega)} - 20 \log k, \tag{37}$$

where $S_{\varphi'}(\omega)$ is given by (34), $1/k^2\ S_{\theta_2}(\omega)$ is given by (22), and $1/k^2\ S_{\theta_3}(\omega)$ is given by (30). A digital computer program has been written which will evaluate (36) and (37) for any values of the transmission deviation coefficients, pre-emphasis coefficients, rms frequency deviation due to the baseband signal, top baseband frequency, and AM/PM conversion factor. The derivations of $\mathfrak{F}[R_{\varphi'}{}^2(\tau)]$, $\mathfrak{F}[R_{\varphi'\varphi''}{}^2(\tau)]$, $\mathfrak{F}[R_{\varphi''}{}^2(\tau)]$, and $\mathfrak{F}[R_{\varphi'}{}^3(\tau)]$ in a form applicable to a digital computer program are given in Appendix II of Ref. 2.

III. NOISE PROPERTIES AND CHARACTERISTICS

The previous material provided the mathematical treatment of AM/PM intermodulation noise. In this section we will document the various properties of both AM/PM intermodulation noise and intermodulation noise due to transmission deviations.* Also, the characteristics of these two noise phenomena will be explored by utilizing a representative system model. Both noise contributors are treated in parallel throughout the section for comparison purposes. The results are presented in three discrete modes: (*i*) properties which are true in general; (*ii*) properties which are approximately true; and (*iii*) characteristics which are derived from a representative system model. The theoretical treatment previously presented was for a transmission medium given by (10). In this section we will confine our analysis to the power series transmission deviations in (10). This is done for two reasons: (*i*) the properties of the two noise phenomena can be concisely documented for power series transmission deviations; and (*ii*) the gain and phase ripple properties need more analysis as well as mathematical treatment in order to fully characterize the effects of ripples in the transmission medium.

* The information for this latter noise contributor was obtained from Ref. 2, which gives it implicitly, as well as from the associated digital computer program.

3.1 *General Properties*

Equation (8) of this paper and (23) of Ref. 2 have been expressed in terms of the transmission deviations and tabulated as shown in Table I. This table is an extension of Table 21-1 of Ref. 4.

TABLE I—AMPLITUDE AND PHASE MODULATION CAUSED BY TRANSMISSION DEVIATIONS

Type of transmission deviation	Resulting amplitude modulation, $a(t)$	Resulting phase modulation, $\varphi_o(t) - \varphi(t)$
Linear gain, g_1	$g_1\varphi' - \frac{1}{2}g_1^2\varphi'^2 + \frac{1}{3}g_1^3\varphi'^3$	
Parabolic gain, g_2	$g_2\varphi'^2 + \frac{1}{2}g_2^2\varphi''^2$	$-g_2\varphi'' + g_2^2\varphi'^2\varphi''$
Cubic gain, g_3	$-g_3\varphi''' + g_3\varphi'^3$	$-3g_3\varphi'\varphi''$
Quartic gain, g_4	$-4g_4\varphi'\varphi''' - 3g_4\varphi''^2$	$g_4\varphi'''' - 6g_4\varphi'^2\varphi''$
Parabolic phase, b_2	$b_2\varphi'' + 2b_2^2\varphi'\varphi''' + b_2^2\varphi''^2$	$-\frac{1}{2}b_2^2\varphi'''' + b_2\varphi'^2$ $+ 2b_2^2\varphi'^2\varphi''$
Cubic phase, b_3	$3b_3\varphi'\varphi''$	$-b_3\varphi''' + b_3\varphi'^3$
Quartic phase, b_4	$-b_4\varphi'''' + 6b_4\varphi'^2\varphi''$	$-4b_4\varphi'\varphi''' - 3b_4\varphi''^2$
Interaction terms	$-[g_1b_3 + g_2b_2]\varphi''''$ $+ [2g_1b_2]\varphi'\varphi'' + g_1g_3\varphi'\varphi'''$ $-g_1g_2\varphi'^3 + [3g_1b_3 + 4g_2b_2$ $- 2g_1^2b_2]\varphi'^2\varphi''$	$-g_1b_2\varphi''' + g_1g_2\varphi'\varphi''$ $- 4(g_1b_3 + g_2b_2)\varphi'\varphi'''$ $-3(g_1b_3 + g_2b_2)\varphi''^2$ $+ (3g_1g_3 - g_1^2g_2)\varphi'^2\varphi''$

Input signal = $\exp\{i[\omega_c t + \varphi(t)]\}$; output signal = $\exp[a(t)]\exp\{i[\omega_c t + \varphi_o(t)]\}$; transmission medium transfer function = $Y(\omega + \omega_c) = [1 + g_1\omega + g_2\omega^2 + g_3\omega^3 + g_4\omega^4]\exp\{i[b_2\omega^2 + b_3\omega^3 + b_4\omega^4]\}$.
The argument of all the amplitude and phase functions is t.

The order of the noise produced by different transmission deviations (e.g., g_1, b_2) are given in Table II for intermodulation noise due to transmission deviations and for AM/PM intermodulation noise. Two rules of thumb can be stated. For intermodulation noise due to transmission deviations the rule is:

Even-order gain and delay transmission deviations cause odd-order noise.

TABLE II — ORDER OF NOISE

Transmission deviation	Intermodulation noise	
	Due to transmission deviations	Due to AM/PM conversion
Linear gain (g_1):	No noise	*Second and third
Parabolic gain (g_2):	Third	Second
Cubic gain (g_3):	Second	Third
Quartic gain (g_4):	Third	Second
Linear delay (b_2):	*Second and third	Second
Parabolic delay (b_3):	Third	Second
Cubic delay (b_4):	Second	Third

* Indicates predominant component of the two possible.

Odd-order gain and delay transmission deviations cause even-order noise.
For AM/PM intermodulation noise the rule is, for those transmission deviations that cause significant relative noise (will become apparent later),

Even-order gain and delay transmission deviations cause even-order noise.

Odd-order gain and delay transmission deviations cause odd-order noise.

The two types of intermodulation noise are related to the magnitude of the transmission deviation coefficient by the relationships shown in Table III. Once a noise response is obtained for a particular system and transmission deviation coefficient value, then the system noise for any other coefficient value typically encountered in transmission systems can be easily predicted.

3.2 *Approximate Properties*

The variation in the top message channel noise, for both noise contributors, with number of channels, assuming the peak frequency deviation remains constant as the number of message channels increase, is shown in Table IV for the different transmission deviations. These approximate relationships yield results with an error of <1 dB for smooth pre-emphasis functions typically used in broadband radio systems.

The assumptions used were that the peak frequency deviation remained constant, and that a typical frequency division multiplex plan was used. The rms frequency deviation, due to the baseband signal,

TABLE III — VARIATION OF RELATIVE NOISE WITH TRANSMISSION DEVIATION COEFFICIENT VALUE

Transmission deviation	Intermodulation noise	
	Due to transmission deviations	Due to AM/PM conversion
Linear gain (g_1):	No noise	*40 log $\lvert g_1'/g_1 \rvert$, 60 log $\lvert g_1'/g_1 \rvert$
Parabolic gain (g_2):	40 log $\lvert g_2'/g_2 \rvert$	$\leqq$20 log $\lvert g_2'/g_2 \rvert$ (approximation error $<\frac{1}{2}$ dB)
Cubic gain (g_3):	20 log $\lvert g_3'/g_3 \rvert$	20 log $\lvert g_3'/g_3 \rvert$
Quartic gain (g_4):	20 log $\lvert g_4'/g_4 \rvert$	20 log $\lvert g_4'/g_4 \rvert$
Linear delay (b_2):	*20 log $\lvert b_2'/b_2 \rvert$, 40 log $\lvert b_2'/b_2 \rvert$	40 log $\lvert b_2'/b_2 \rvert$
Parabolic delay (b_3):	20 log $\lvert b_3'/b_3 \rvert$	20 log $\lvert b_3'/b_3 \rvert$
Cubic delay (b_4):	20 log $\lvert b_4'/b_4 \rvert$	20 log $\lvert b_4'/b_4 \rvert$

Where the prime (′) notation depicts the terminal value and the unprimed notation indicates the initial value.
* Indicates predominant component of the two possible.

TABLE IV — VARIATION OF TOP CHANNEL NOISE WITH NUMBER OF MESSAGE CHANNELS

Transmission deviation	Intermodulation noise: Due to transmission deviations	Intermodulation noise: Due to AM/PM conversion
Linear gain (g_1):	No noise	$\cong 20 \log N'/N$
Parabolic gain (g_2):	Relative *top* channel noise increase $\cong 41 \log N'/N$	$\cong 21 \log N'/N$
Cubic gain (g_3):	Relative *top* channel noise increase $\cong 39 \log N'/N$	$\cong 21 \log N'/N$
Quartic gain (g_4):	Relative *top* channel noise increase $\cong 41 \log N'/N$	$\cong 57 \log N'/N$
Linear delay (b_2):	Relative *top* channel noise increase $\cong 21 \log N'/N$	$\cong 58 \log N'/N$
Parabolic delay (b_3):	Relative *top* channel noise increase $\cong 23 \log N'/N$	$\cong 39 \log N'/N$
Cubic delay (b_4):	Relative *top* channel noise increase $\cong 58 \log N'/N$	$\cong 40 \log N'/N$

Where N' = increased number of channels; N = initial number of channels.

was allowed to change, accordingly, as the number of message channels increased.

3.3 *Noise Characteristics*

3.3.1 *Representative System Model*

As a system model, we will use the following system parameters:

N = number of message channels = 1200

f_b = top baseband frequency = 5.772 MHz

ΔF = peak frequency deviation = 4 MHz

σ = rms frequency deviation due to the multichannel baseband signal = 0.771 MHz.

The pre-emphasis characteristic is shown in Fig. 5 and can be expressed by

$$| P(\omega) |^2 = 0.9989 + 3.5839 \times 10^{-1} f^2 - 5.0245 \times 10^{-3} f^4 + 3.894 \times 10^{-5} f^6,$$

where f is in MHz.

3.3.2 *Noise Response for the Individual Transmission Deviations*

It is instructive to show the individual noise responses on a comparative basis. This can be done by letting all gain transmission deviations

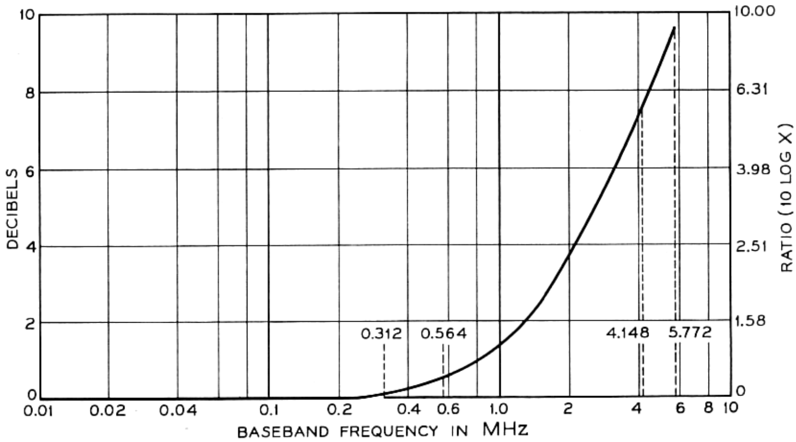


Fig. 5 — Pre-emphasis characteristic.

have 1-dB distortion, relative to the carrier, at 10 MHz away from the carrier. Also, we let all delay transmission deviations have 1 nanosecond (ns) distortion, relative to the carrier, at 10 MHz away from the carrier. This allows us to directly compare the noise contributions of the different gain and phase transmission deviations, respectively, and also allows for some sort of pseudo comparison between a 1-dB gain distortion and a 1-ns delay distortion. The intermodulation noise response, due to transmission deviations, for the different transmission deviations are shown in Fig. 6. Similarly, the AM/PM intermodulation noise responses are shown in Fig. 7. Note that the responses in Fig. 7 are for $k = 1.0$ radian or a 6.6 degrees/dB AM/PM conversion device. For any other value of k, say k_1, we raise or lower the responses according to 20 log k_1, as indicated by (36) and (37).

It is interesting to note that linear delay is an important contributor to intermodulation noise, due to transmission deviations, but is not a significant AM/PM intermodulation noise contributor. Also, we observe that parabolic gain is a large relative contributor for AM/PM intermodulation noise but is a negligible relative contributor for intermodulation noise due to transmission deviations. As a side point, we point out that parabolic gain is also a significant source of baseband amplitude distortion.

The phase transmission deviation noise responses in Figs. 6 and 7 are of particular interest because the values used in these two figures are realistic even for an equalized system; this is not the case for the

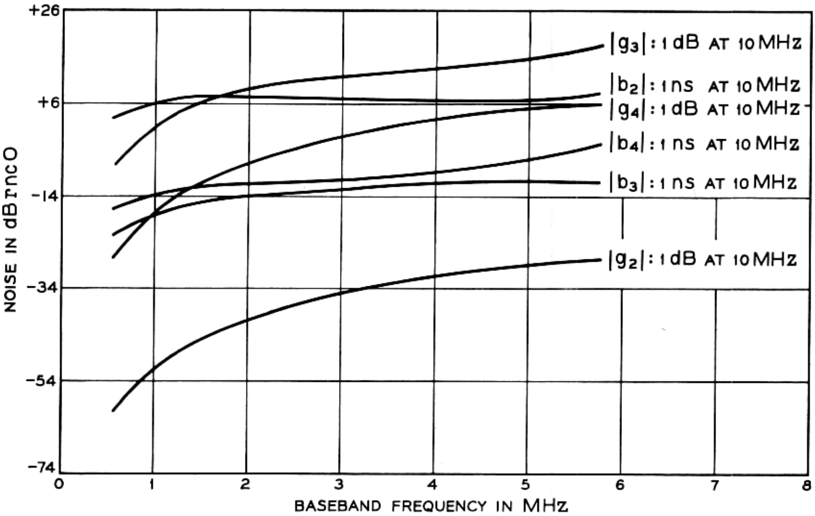


Fig. 6 — Intermodulation noise due to transmission deviations.

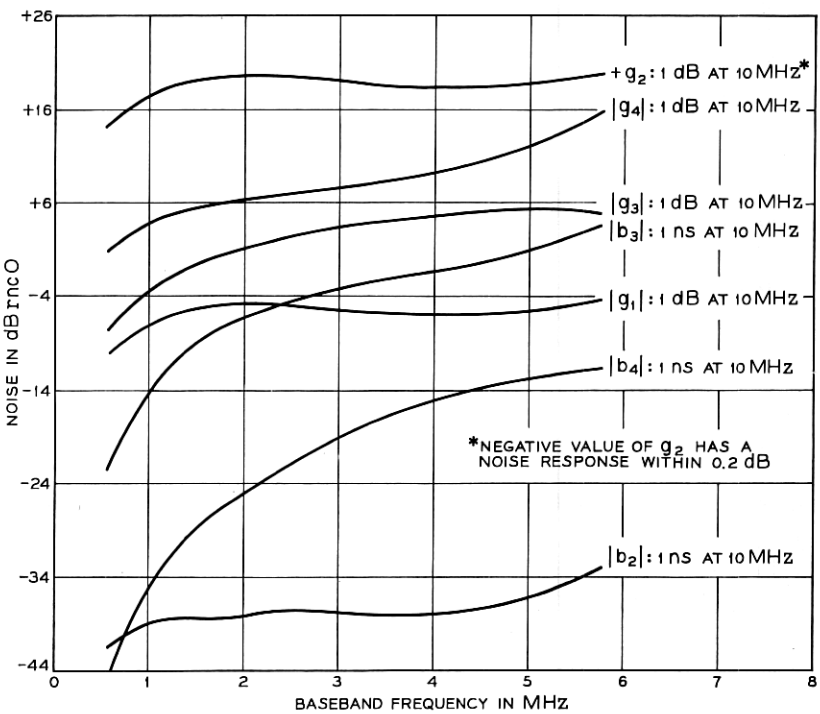


Fig. 7 — AM/PM intermodulation noise ($k = 1.0$ radian).

gain transmission deviation values used so one need not be unduly alarmed at first inspection of the noise responses shown. However, the gain deviation noise responses are of interest in order to determine which types of gain deviations a particular noise source is most sensitive to.

Comparison of Figs. 6 and 7 and Table II show that for all transmission deviations of significance, the order of the noise distortions are different for the two noise phenomena. Coupling this with the results of the Appendix shows that for a given transmission deviation, the two noise responses are uncorrelated.

Of great significance and importance is the parabolic delay AM/PM intermodulation noise response. We see from Fig. 7 that this particular delay deviation is, by far, the largest contributor of AM/PM intermodulation noise compared to the other delay terms. The importance of this finding lies in the fact that TWT amplifiers, when used as output power tubes in broadband radio systems, are separated from transmitter modulators (used to go from IF to RF) by band pass filters which may possess large amounts of parabolic delay. Hence, we have large parabolic delay distortion prior to an important AM/PM conversion device. The noise impairment due to this typical system arrangement will be examined in a later section.

Another point of interest is the noise response for linear gain. We see that linear gain is not a significant AM/PM intermodulation noise contributor. This is useful knowledge because in the past system requirements for linear gain have been set based on speculated AM/PM intermodulation noise impairments, as well as on derivable baseband amplitude distortion due to linear gain and AM/PM conversion.

3.3.3 *Effects of Interaction Terms*

Referring back to Table I we note the row marked interaction terms. By the form of the terms involved it is obvious why they are so named. If one were to evaluate (22) and (30) in terms of the transmission deviations explicitly, he would find that over 80 percent of the terms are interaction terms. To examine the effects of these interaction terms we compare the response we would get if we combined the curves shown in Fig. 6, for example, on a power basis with the response we would obtain by using all the transmission deviations at once, i.e., by taking into account the interaction terms. There are a large number of possibilities that could be examined, but to put the problem in perspective the analysis considered only the cases shown in Table V. The results for the two noise phenomena are shown in Figs. 8 and 9. The responses

TABLE V — CASES CONSIDERED IN THE STUDY OF INTERACTION EFFECTS

Case	Condition*
1	Power addition of noise responses due to individual transmission deviations (all g's and b's positive)
2	Noise response under the condition: g_1 negative and all other g's and b's positive
3	Noise response under the condition: g_2 negative and all other g's and b's positive
4	Noise response under the condition: g_3 negative and all other g's and b's positive
5	Noise response under the condition: g_4 negative and all other g's and b's positive
6	Noise response under the condition: b_2 negative and all other g's and b's positive
7	Noise response under the condition: b_3 negative and all other g's and b's positive
8	Noise response under the condition: b_4 negative and all other g's and b's positive
9	Noise response under the condition: all g's and b's positive
10	Noise response under the condition: all g's and b's negative

$$
\begin{aligned}
\text{Where } \mid g_1 \mid &= 1 \text{ dB at } 10 \text{ MHz} \\
\mid g_2 \mid &= 1 \text{ dB at } 10 \text{ MHz} \\
\mid g_3 \mid &= 1 \text{ dB at } 10 \text{ MHz} \\
\mid g_4 \mid &= 1 \text{ dB at } 10 \text{ MHz} \\
\mid b_2 \mid &= 1 \text{ ns at } 10 \text{ MHz} \\
\mid b_3 \mid &= 1 \text{ ns at } 10 \text{ MHz} \\
\mid b_4 \mid &= 1 \text{ ns at } 10 \text{ MHz}
\end{aligned}
$$

* All the conditions take into account the effects of interaction terms except for case 1.

shown in Fig. 8 are rewarding from a systems analysis standpoint because it indicates that the interaction components for intermodulation noise, due to transmission deviations, do not significantly perturb the noise response obtained by adding up the individual transmission deviation noise responses on a power basis. Hence, for this noise source, a system analyst could set requirements based on power addition of the individual noise responses and be fairly confident that the actual system noise response, due to transmission deviations, will be within a dB of that response.

We see from Fig. 9 that the above desirable property does not hold for AM/PM intermodulation noise. The responses shown in Fig. 9 deviate significant amounts from the power addition response (case 1) by mere shifts of signs, the greatest departures occurring for parabolic and quartic gain distortion which are, in their own right, the largest relative noise contributors as evident from Fig. 7. The relative tendencies indicated in Fig. 9 also occur when typical equalized repeater trans-

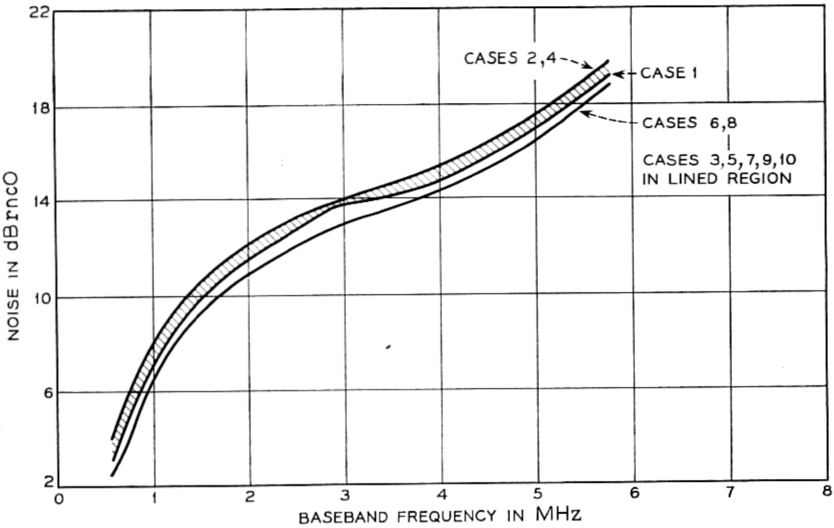


Fig. 8 — Intermodulation noise due to transmission deviations — effects of interaction terms (refer to Table V for case listing).

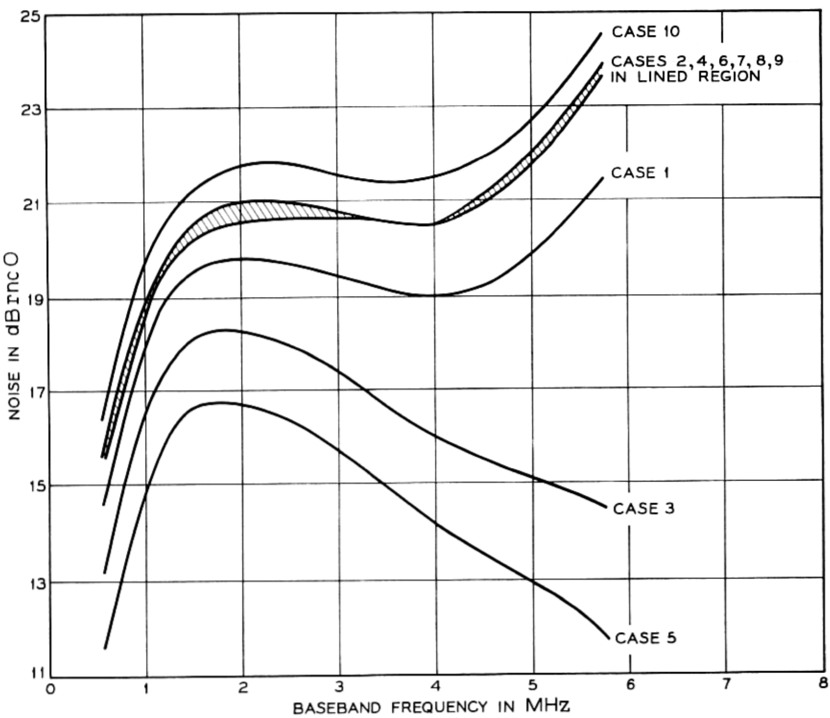


Fig. 9 — AM/PM intermodulation noise — effects of interaction terms (k = 1.0 radian) (refer to Table V for case listing).

mission deviation values are used. In fact, least squares approximations on equalized repeaters, RF band pass filters, etc., yield values for parabolic and quartic gain which are either, or both, negative, i.e., a loss with increasing frequency. Hence, even under practical situations one cannot, in general, expect power addition of the individual transmission deviation AM/PM intermodulation noise responses to yield representative AM/PM intermodulation noise performance.

3.3.4 *Noise Response for a Representative Radio System Repeater*

The results up to this point utilized representative system parameters, but normalized values for the transmission deviations were used. Of interest is the predicted noise response for a typical situation, i.e., making use of values typically encountered in practice. We will now use the representative gain and delay responses shown in Fig. 10 for an unequalized and equalized radio repeater. The predicted intermodulation noise responses, due to transmission deviations, are shown in Fig. 11. It is obvious that the equalization has greatly improved the system's noise response.

To examine the AM/PM intermodulation noise we take note of the previously mentioned fact that the TWT has a band pass filter (whose gain and delay responses are given in Fig. 10) preceding it. The AM/PM intermodulation noise due to the band pass filter and the TWT amplifier (assuming 2.5 degrees/dB) is also shown in Fig. 11.

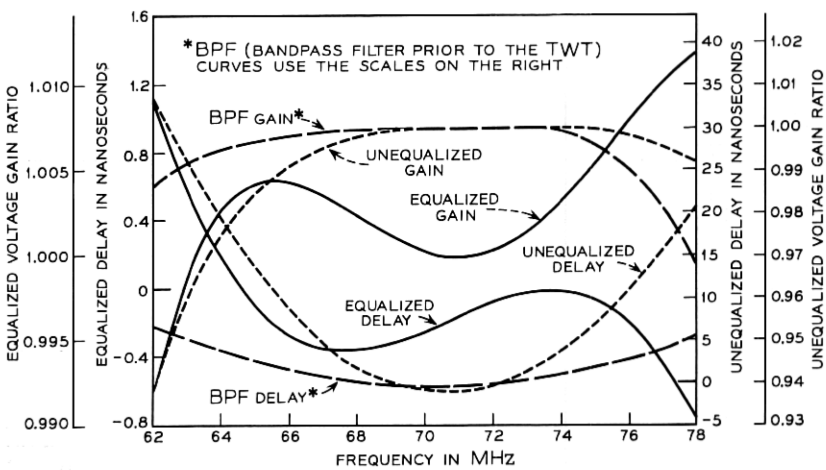


Fig. 10 — Gain and delay characteristics.

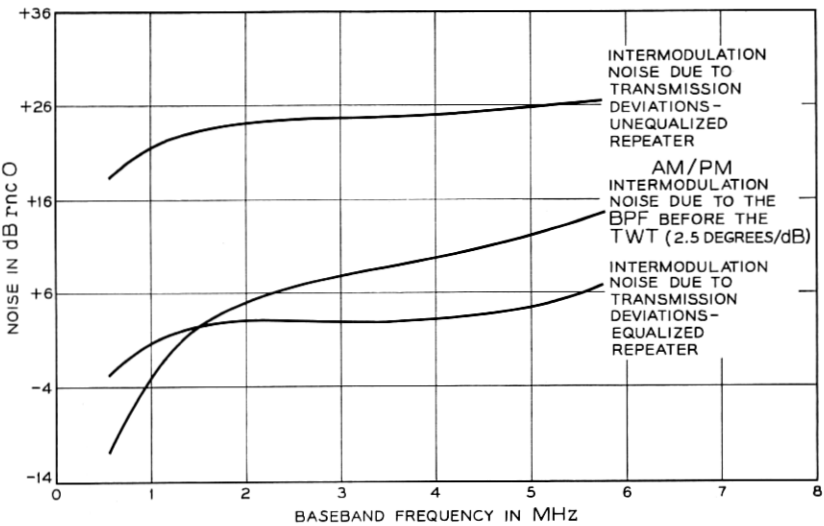


Fig. 11 — Representative radio system noise response.

We see from Fig. 11 that the AM/PM intermodulation noise in the top channel is much larger than the intermodulation noise due to transmission deviations for an equalized repeater. The repeater equalizer is designed to correct for gain and delay shapes obtained from measurements which do not recognize the AM/PM conversion phenomenon. Hence, repeater equalizers based on such measurements, even though effective for reducing intermodulation noise due to transmission deviations, prove ineffective for AM/PM intermodulation noise which occurs as indicated. In other words, the system has an AM/PM intermodulation noise floor which is transparent to external gain and delay measurements.

The transmission deviation of the band pass filter which is the major noise contributor is the parabolic delay term. Hence, to reduce the AM/PM intermodulation noise one must devise some method of correcting for this transmission deviation. Two means of equalizing the band pass filter are: (*i*) pre-equalization at IF prior to up-converting in the transmitter modulator; and (*ii*) microwave equalization directly before or after the bandpass filter. The first method may not yield perfect correction because up-converters using varactor diodes possess AM/PM conversion characteristics, in some cases 1 degree/dB. In essence, it would effectively be like trading noise due to 2.5 degrees/dB for noise due to 1.0 degrees/dB or an 8-dB improvement in the ideal case, i.e., no compression in the up-converter and a perfect inverse band pass filter characteristic. However, an improvement anywhere near this value would greatly reduce the effects of AM/PM conversion.

IV. CONCLUSIONS

Two noise contributors in FM systems are: (*i*) intermodulation noise due to transmission deviations; and (*ii*) intermodulation noise due to transmission deviations and AM/PM conversion. This latter source of noise is designated "AM/PM intermodulation noise" in this paper. Analysis was carried out in order to predict the second- and third-order AM/PM intermodulation noise for the transmission medium given in (10) and a continuous pre-emphasis function. Flat Gaussian noise was used to simulate the unpre-emphasized baseband signal so the results are consistent with the laboratory system tests using "noise loading". Expressions were derived which specify the signal-to-noise ratio in terms of system parameters, transmission deviations, pre-emphasis characteristics and AM/PM conversion parameter. The latter parameter, assumed to be a real constant, was separated from the body of the calculations so that the resulting noise responses could be easily altered for any value of AM/PM conversion.

The paper presented general noise properties and characteristics for the two noise contributors. This material was presented in parallel, for the two noise contributors, for comparison purposes. The order of the noise component for different transmission deviations was given so that one would know if a given transmission deviation causes second- or third-order noise. The variation of the relative noise with transmission deviation coefficient value was given so that a system analyst can determine the relative detriment to a system response that would result from a change in a given transmission deviation. Another useful result was the variation of top channel noise with number of message channels. This would be of use in the case where one is interested in increasing a system's message channel capacity.

Noise responses were given using a representative radio system model. It was found when all gain transmission deviations had the same distortion and when all delay transmission deviations had the same distortion that: (*i*) for intermodulation noise due to transmission deviations the cubic and quartic gain terms created the greatest top channel noise due to gain transmission deviations, and that linear delay created the greatest top channel noise due to delay transmission deviations; and (*ii*) for AM/PM intermodulation noise the parabolic and quartic gain terms created the greatest top channel noise due to gain transmission deviations, and that parabolic delay created the greatest top channel noise due to delay transmission deviations. The effects of interaction terms were examined. It was found that interaction terms do not significantly perturb the noise response from that of the case of power addition of the individual noise responses for intermodulation noise due to transmission deviations. However, this desirable property did not hold for AM/PM intermodulation noise which says that power addition of the individual noise responses may be in gross error; in other words, interaction terms must be considered when evaluating AM/PM intermodulation noise.

The intermodulation noise due to both noise contributors was predicted for a representative radio system repeater. It was observed that the AM/PM intermodulation noise due to the band pass filter preceding the TWT amplifier created more top channel noise than that due solely to the equalized transmission characteristic, i.e., intermodulation noise due to transmission deviations. Possible correction methods were given.

A point of interest, is that the two noise contributors considered in this paper are correlated so that combining the two spectra together assuming random addition, i.e., power addition, is not sufficient in general. The significance of this correlation is presently being examined and will be reported on in a later paper.

V. ACKNOWLEDGMENT

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APPENDIX

Uncorrelated Second- and Third-Order Distortions

We will show here that the second-order distortion, $\theta_2(t)$, and the third-order distortion, $\theta_3(t)$, are uncorrelated. Consider

$$\theta_T(t) = \theta_2(t) + \theta_3(t). \tag{38}$$

Now the autocorrelation function of $\theta_T(t)$ is

$$R_{\theta_T}(\tau) = R_{\theta_2}(\tau) + R_{\theta_3}(\tau) + R_{\theta_2\theta_3}(\tau) + R_{\theta_3\theta_2}(\tau) \tag{39}$$

where, e.g., $R_{\theta_2\theta_3}(\tau)$ is the cross-correlation function of $\theta_2(t)$ and $\theta_3(t)$. Taking the Fourier transform of (39) gives

$$S_{\theta_T}(\omega) = S_{\theta_2}(\omega) + S_{\theta_3}(\omega) + S_{\theta_2\theta_3}(\omega) + S_{\theta_3\theta_2}(\omega). \tag{40}$$

From Fig. 2, we have

$$\theta_2(t) = x(t) + y(t) \tag{41}$$

so it follows that

$$R_{\theta_2\theta_3}(\tau) = R_{x\theta_3}(\tau) + R_{y\theta_3}(\tau). \tag{42}$$

Referring to Figs. 2 and 3 we have, using the relationship for the cross-correlation of linearly transformed random functions,[3]

$$S_{\theta_2\theta_3}(\omega) = G_1(-\omega)\, G_3(\omega)\, S_{\varphi'^2\varphi'^3}(\omega) + G_2(-\omega)\, G_3(\omega)\, S_{\varphi''^2\varphi'^3}(\omega). \tag{43}$$

Now we can write

$$S_{\varphi'^2\varphi'^3}(\omega) = \mathfrak{F}\,[R_{\varphi'^2\varphi'^3}(\tau)] = \mathfrak{F}\,[\text{ave}\,(\varphi'^2\varphi'^3)] \tag{44}$$

and

$$S_{\varphi''^2\varphi'^3}(\omega) = \mathfrak{F}\,[R_{\varphi''^2\varphi'^3}(\tau)] = \mathfrak{F}\,[\text{ave}\,(\varphi''^2\varphi'^3)], \tag{45}$$

where $\mathfrak{F}$ stands for the Fourier Transform.

The phase modulating signal, $\varphi(t)$ represents the multichannel message load and so for a large number of talkers $\varphi(t)$ is Gaussian with zero mean.[5] It follows that derivatives of $\varphi(t)$ are Gaussian with zero mean. It can be shown that[6]

$$\text{ave}\,[x_1^{r_1} \cdots x_n^{r_n}] = 0, \qquad \sum_{i=1}^{n} r_i \text{ odd} \tag{46}$$

where $x_1 \cdots x_n$ are Gaussian random variables with zero mean, and $r_1 \cdots r_n$ are any set of integers. Hence, letting

$$x_1 = \varphi'$$

$$x_2 = \varphi'$$

in (44), and letting

$$x_1 = \varphi''$$

$$x_2 = \varphi'$$

in (45) gives, using (46),

$$S_{\theta_2\theta_3}(\omega) = 0.$$

Similarly,

$$S_{\theta_3\theta_2}(\omega) = 0.$$

Hence, $\theta_2(t)$ and $\theta_3(t)$ are uncorrelated so

$$S_{\theta_T}(\omega) = S_{\theta_2}(\omega) + S_{\theta_3}(\omega).$$

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